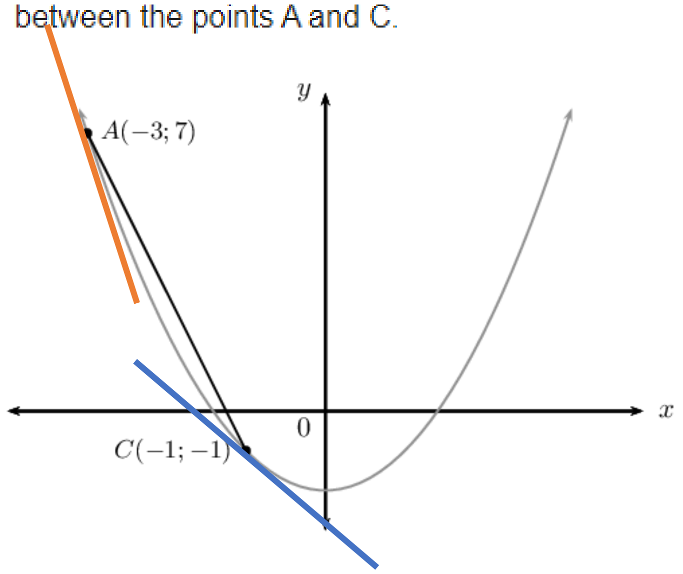
Revision Webinar # 9

Calculus Part 1: Average Gradient, Gradient at a point including first principles and rules for differentiation.

Calculus Part 2: Equation of tangent, Cubic graphs, optimization problems

Average Gradient of a curve between two points.

The **average gradient** between any two points on a curve is the **gradient** of the straight line passing through the two points. This is the **average gradient** of the curve between the points A and C.



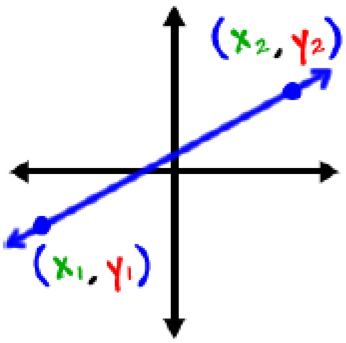
Average Gradient = $\frac{7-(-1)}{-3-(-1)} = \frac{8}{-2} =$

Six Ways to Determine the Gradient of a straight line

(1) Given Equation

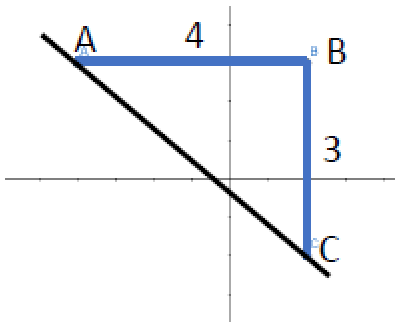
$$ax + by + c = 0 \rightarrow y = -\frac{a}{b}x - \frac{c}{a} \rightarrow m = -\frac{a}{b}$$

(2) Given Two points on a line



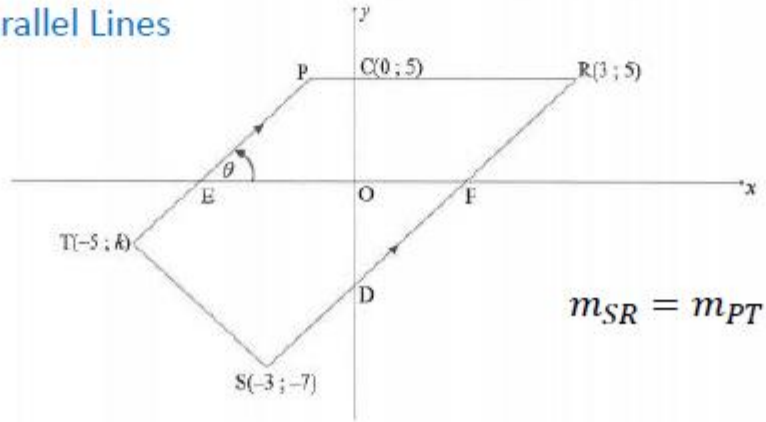
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(3) Given vertical and horizontal distance



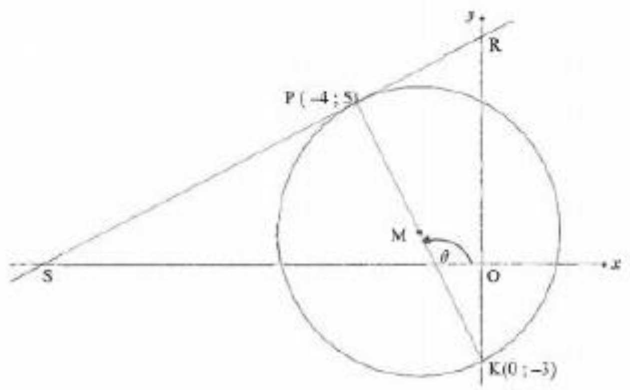
$$m = -\frac{3}{4}$$

(4) Parallel Lines



$m_{SR} = m_{PT}$

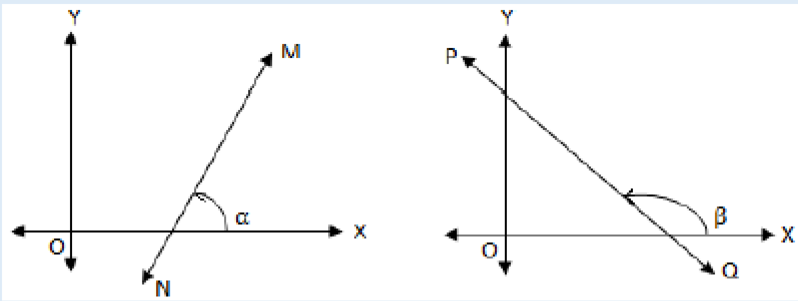
(5) Perpendicular Lines



$S\hat{P}K = 90°$; *radius* $\perp$ *tangent*

$\therefore m_{SR} \times m_{PK} = -1$

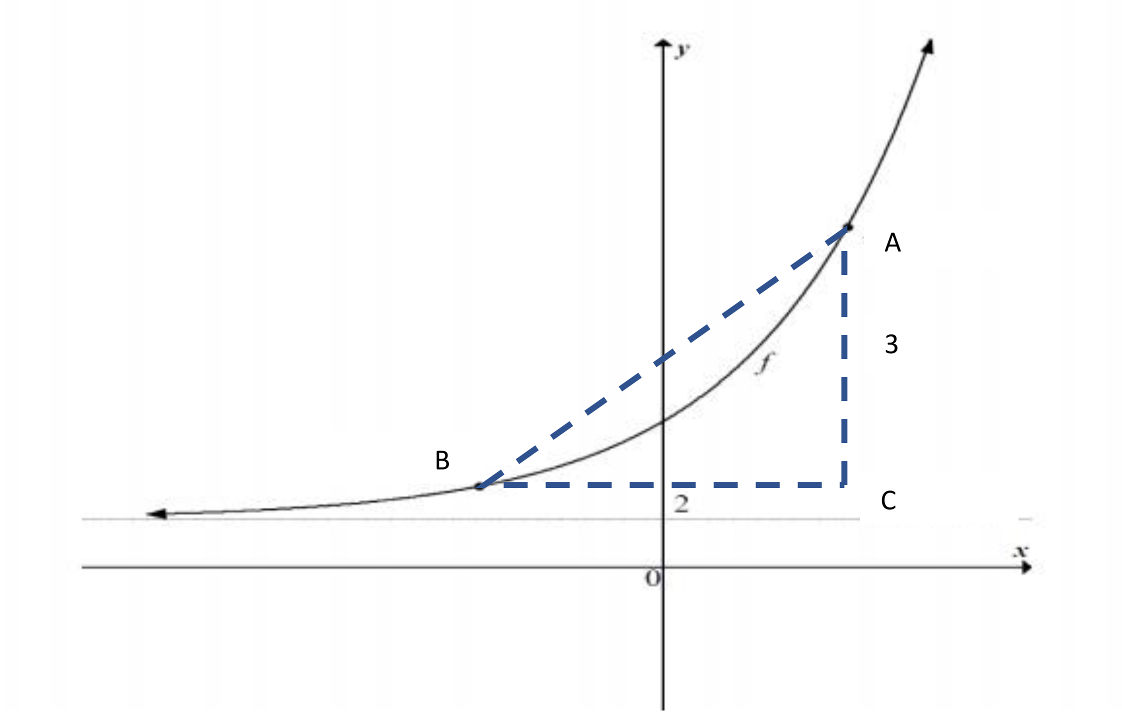
(6) Given Angle of Inclination



$m = \tan\alpha > 0$ $\qquad$ $m = \tan\beta < 0$

Examples

1. Determine the average gradient of $f(x) = x^2 - 4x$ between the points where *x=2* and *x=4*.
2. A straight line with equation *2y-x=5* passes through two points A and B on a curve g. Determine the average gradient of g between A and B.
3. The graph of *f* is sketched below.
   AC = 3 units and BC = 2 uints



   Determine the average gradient of f between A and B.

4. The line passing through A and B on a curve g is perpendicular to the line 3y+4x=5. Determine the average gradient of g between A and B.

Solutions

AVERAGE GRADIENT

EXAMPLES

1. $(2; -4)$ $(4; 0)$

$y = 2^2 - 4(2)$ $y = 4^2 - 4(4)$

$y = -4$ $y = 0$

$\therefore$ Average gradient $= \frac{0-(-4)}{4-2} = 2$

2. $2y - x = 5$

$2y = x + 5$

$y = \frac{1}{2}x + \frac{5}{2}$

$\therefore$ Average gradient of g between A and B is $\frac{1}{2}$.

3. Average gradient of f between A and B is $\frac{3}{2}$.

4. $3y = -4x + 5$

$y = -\frac{4}{3}x + \frac{5}{3}$

$m_{AB} = \frac{3}{4}$

$\therefore$ Average gradient of g between A and B is $\frac{3}{4}$.

First Principles- Gradient at a point.

Notation:

$f^{/}(x)$ ***represents the gradient(derivative) of f at the point*** $(x; f(x))$

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Using the definition (first principle), find the derivative, $f'(x)$ for a, b and c constants:

(a) $f(x) = ax^2 + bx + c$;

(b) $f(x) = ax^3$;

(c) $f(x) = \frac{a}{x}$;

(d) $f(x) = c$

Examples

1.

Differentiate $f(x) = x^2$ from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left((x+h)^2\right) - \left(x^2\right)}{h}$$ Substitute for f(x) and f(x+h)

$$= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2\right) - \left(x^2\right)}{h}$$ Multiply the brackets

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$ Simplify

$$= \lim_{h \to 0} 2x + h$$ Divide by h

$$= 2x$$ Apply the limit, i.e. make $h = 0$

2. Determine the gradient of $f(x) = x^2 + 2$ at the point where x = 3.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left((x+h)^2 + 2\right) - \left(x^2 + 2\right)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 + 2 - x^2 - 2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x + h)}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x \end{aligned}$$

Therefore the gradient of f at the point where x = 3 is 2(3)=6

3. Determine $f^{/}(x)\ if\ f(x) = -2x^3$

$$f(x) = -2x^3$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{-2(x+h)^3 - (-2x^3)}{h} \\ &= \frac{-2(x+h)(x^2 + 2xh + h^2) + 2x^3}{h} \\ &= \frac{-2(x^3 + 3x^2h + 3xh^2 + h^3) + 2x^3}{h} \\ &= \frac{-2x^3 - 6x^2h - 6xh^2 - 2h^3 + 2x^3}{h} \end{aligned}$$

$$= \frac{-2x^3 - 6x^2h - 6xh^2 - 2h^3 + 2x^3}{h}$$

$$= \frac{-6x^2h - 6xh^2 - 2h^3}{h}$$

$$= \frac{h(-6x^2 - 6xh - 2h^2)}{h}$$

$$f'(x) = \lim_{h \to 0} (-6x^2 - 6xh - 2h^2)$$

$$= -6x^2$$

4. Given: $g(x) = \frac{-2}{x}$

   4.1 Determine $\frac{g(1+h)-g(1)}{h}$

   4.2 What does your answer represent in 4.1?

   4.3 Determine $\lim_{h \to 0} \frac{g(1+h)-g(1)}{h}$

   4.4 What does your answer represent in 4.3?

4.1 $\frac{g(1+h) - g(1)}{h}$

$= \frac{1}{h}\left(\frac{-2}{1+h} - \frac{-2}{1}\right)$

$= \frac{1}{h}\left(\frac{-2 + 2(1+h)}{1+h}\right)$

$= \frac{1}{h}\left(\frac{-2+2+2h}{1+h}\right)$

$= \frac{1}{h}\left(\frac{2h}{1+h}\right)$

$= \frac{2}{1+h}$

4.2 Average gradient of g between $(1; g(1))$ and $(1+h; g(1+h))$.

4.3 $\lim_{h \to 0} \frac{2}{1+h} = 2$.

4.4 gradient of g at $(1; g(1))$.

5.

Differentiate $g(x) = \frac{1}{4}$ from first principles and interpret the answer.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{\frac{1}{4} - \frac{1}{4}}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0 \end{aligned}$$

The gradient of $g(x)$ is equal to 0 at any point on the graph. The derivative of this constant function is equal to 0.

Differentiating using rules

Notation

$\frac{dy}{dx}$ represents the gradient(derivative) of y=f(x) at the point (x;f(x))

$\frac{d}{dx}(\boldsymbol{f}(\boldsymbol{x})$ represents the gradient(derivative) of f(x) at the point (x;f(x))

Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Examples

1. Determine the derivative with respect to x for each of the following:

1.1 $f(x) = 4x^{20} - \sqrt[4]{x^3} + \pi^3 + 6$

1.2 $y = \frac{9x^2-1}{3x^2-x}$

1.3 $\sqrt{2x} - \frac{x}{\sqrt{x}}$

2.

Use differentiation rules to do the following:

2.1 Determine $f'(x)$ if $f(x) = (x+2)^2$

2.2 Determine $f'(x)$ if $f(x) = \frac{(x+2)^3}{\sqrt{x}}$

2.3 Determine $\frac{dy}{dt}$ if $y = \frac{t^2 - 1}{2t + 2}$

2.4 Determine $f'(\theta)$ if $f(\theta) = (\theta^{3/2} - 3\theta^{-1/2})^2$

Solutions

DIFFERENTIATING USING RULES

EXAMPLES

1.1 $f(x) = 4x^{20} - x^{3/4} + \pi^3 + 6$

$f'(x) = 80x^{19} - \frac{3}{4}x^{-\frac{1}{4}}$

1.2 $y = \frac{(3x-1)(3x+1)}{x(3x-1)} = \frac{3x+1}{x}$

$\therefore\ y = \frac{3x}{x} + \frac{1}{x}$

$\therefore\ y = 3 + x^{-1}$

$\therefore\ \frac{dy}{dx} = -x^{-2}$

1.3 $\frac{d}{dx}\left(\sqrt{2x} - \frac{x}{\sqrt{x}}\right)$

$= \frac{d}{dx}\left(\sqrt{2}.\sqrt{x} - \frac{x}{x^{\frac{1}{2}}}\right)$

$= \frac{d}{dx}\left(\sqrt{2}\,x^{\frac{1}{2}} - x^{\frac{1}{2}}\right)$

$= \frac{\sqrt{2}}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}.$

2.1

$$f(x) = x^2 + 4x + 4$$
$$f'(x) = 2x + 4$$

2.2

$$f(x) = \frac{(x^2 + 4x + 4)(x + 2)}{\sqrt{x}}$$
$$f(x) = \frac{x^3 + 6x^2 + 12x + 8}{\sqrt{x}}$$
$$f(x) = \frac{x^3}{x^{\frac{1}{2}}} + \frac{6x^2}{x^{\frac{1}{2}}} + \frac{12x}{x^{\frac{1}{2}}} + \frac{8}{x^{\frac{1}{2}}}$$
$$f(x) = x^{\frac{5}{2}} + 6x^{3/2} + 12x^{1/2} + 8x^{-1/2}$$
$$\therefore f'(x) = \frac{5}{2}x^{3/2} + 9x^{1/2} + 6x^{-1/2} - 4x^{-3/2}$$

2.3

$$y = \frac{(t-1)(t+1)}{2(t+1)} = \frac{t-1}{2} = \frac{t}{2} - \frac{1}{2}$$
$$\therefore \frac{dy}{dt} = \frac{1}{2}$$

2.4

$$f(\theta) = \theta^3 - 6\theta + 9\theta^{-1}$$
$$\therefore f'(\theta) = 3\theta^2 - 6 - 9\theta^{-2}$$